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Abstract: In this paper, we focus on extended Kantorovich norms, which seem to be much more flexible than many other "metrics" deriving from optimal mass transport problems. General theoretical results are recalled, and the discrete case is investigated, especially from a computational point of view. An efficient algorithm is then introduced to solve the related linear programming problem, and possible applications are discussed.

Key-words: Kantorovich norm, optimal transportation, numerical résolution.

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Les normes de Kantorovich : un outil d'optimisation.

Résumé : Cet article s'intéresse aux normes de Kantorovich étendues, qui semblent plus flexibles que les autres métriques dérivant de problèmes de transport de masse. Des résultats théoriques généraux sont rappelés, et le cas discret est étudié. Un algorithme rapide est ainsi introduit pour résoudre le problème de programmation linéaire associé au calcul des normes de Kantorovich étendues en dimension 1. Différents domaines d'application sont ensuite abordés.

Mots-clés : norme de Kantorovich, transport optimal, résolution numérique.

1 Introduction

The optimal mass transport problem has been first considered by Monge in 1781 in his "Mémoire sur la théorie des déblais et des remblais" [22]. Few results were known until 1942 and the major advance of Kantorovich [17], whose relaxation of the original problem allowed to use classical mathematical tools. Many theoretical questions remain open, but current works of a growing mathematical community (Brenier, Caffarelli, McCann, Gangbo, Evans...) allow to dream of a clean optimal transport theory. As a bi-product of this theoretical interest, numerous works have been devoted to applications of mass transport techniques in the field of applied mathematics, leading (amongst other things) to original results for some meteorological equations ([4],[8]) and the Fokker-Planck equation [16]. Other applications can be found in [9], where collapsing sand-piles are considered, and in the book of Rachev and Ruschendorf [23], where applications in probability theory are discussed. In each of these problems, mass transportation is closely related to the intrinsic nature of the equations. However, some generic properties of "mass transport induced metrics" (some of them will be recalled in the following) lead to think of a powerful optimization tool. In particular, shape recognition seems to be a stimulating field of application ([11],[10]). Finally, we should mention a practical attempt to use the Wasserstein metric in the field of seismic inversion [21], which points out a major drawback of mass transportation techniques : two signals under comparison should be non-negative and have the same mass. The motivation of this work is then to consider extended Kantorovich norms, introduced in [18] and in [19] in order to allow a transport-style comparison of two finite measures. Removing the assumptions of non-negativity and mass conservation, those norms are much more flexible than the Kantorovich norm or the Wasserstein metric. Now the end of the introduction is devoted to the presentation of the Kantorovich norm and its extensions.

1.1 The Kantorovich norm

Let (K, d) be a separable metric space and $\mathcal{M}(K)$ the set of finite measures on K. We take the following definitions (where μ stands for a measure in $\mathcal{M}(K)$).

- $\mathcal{M}^0(K)$ is the subspace of measures $\mu \in \mathcal{M}(K)$ such that $\mu(K) = 0$,
- $\mathcal{M}_+(K)$ is the set of all non-negative measures on $\mathcal{M}(K)$,
- $supp(\mu)$ is the topological support of μ,
- μ_+ is the positive variation of μ,
- μ_- is the negative variation of μ,
- $|\mu|$ is the total variation of μ,
- $Var(\mu) := |\mu|(K)$.

Theorem 1.1. *For any measure $\mu \in \mathcal{M}^0(K)$, the family Ψ_μ of non-negative measures $\psi \in \mathcal{M}_+(K \times K)$ such that for any borel set e,*

$$\psi(K \times e) - \psi(e \times K) = \mu(e) , \tag{1}$$

is not empty. Define the functional

$$\|.\|_d^0 : \mu \mapsto \inf_{\psi \in \Psi_\mu} \int_{K \times K} d(x, y) d\psi(x, y) , \tag{2}$$

and denote by $\mathcal{M}_d^0(K)$ the subset of measures μ in $\mathcal{M}^0(K)$ such that $\|\mu\|_d^0 < \infty$. Then $\|.\|_d^0$ is a norm on $\mathcal{M}_d^0(K)$.

>From now on, the norm defined in theorem 1.1 will be referred to as the Kantorovich norm. In [13], the author notices that the set Ψ_μ can be replaced by its subset $\hat{\Psi}_\mu$ consisting of measures ψ such that for any borel set e in K,

$$\psi(e \times K) = \mu_-(e) \ , \tag{3}$$

$$\psi(K \times e) = \mu_+(e) \ . \tag{4}$$

Hence we can assume that for any transport plan ψ, we have

$$supp(\psi) \subset supp(\mu_-) \times supp(\mu_+) \ , \tag{5}$$

and the following lemma is obvious.

Lemma 1.2. *For any μ in $\mathcal{M}_d^0(K)$, we have*

$$\|\mu\|_d^0 \leq \frac{1}{2} diam(supp(\mu)) Var(\mu) \ . \tag{6}$$

Economical interpretation

Let e_1 and e_2 be two borel sets. The value $\psi(e_1 \times e_2)$ can be interpreted as the amount of goods (or mass for Monge) carried from the suppliers located in e_1 to the demanders located in e_2.

2 Extended Kantorovich norms

In this section, we introduce the theoretical basic results on extended Kantorovich norms. This presentation follows the one of Hanin [13], to which we refer for further properties and explanations. As we have seen in the previous subsection, the Kantorovich norm is much less sensitive to translation than (e.g.) the total variation. As a consequence, this norm is well suited for the comparison of probability densities. However, $\|.\|_d^0$ is only defined on the subspace $\mathcal{M}_0(K)$, and requires therefore strong assumptions on the data to be used in applications. It is then natural to think about possible extensions of this norm. To our knowledge, two slightly different norms have been introduced, the first (which we denote by the K-norm) by Kantorovich himself, and the second (which we denote by the H-norm) by Hanin.

2.0.1 The K-norm

In this section, we assume furthermore that space K is compact and has a finite diameter. In this case, Kantorovich and Rubinstein first extended the Kantorovich norm. Indeed, introducing some "waste function" p satisfying the following properties

$$p(x) > \sup_{y \in K} |x - y| \ , \tag{7}$$

$$|p(x) - p(y)| \leq |x - y| \ , \tag{8}$$

then the functional defined as

$$\|\mu\|_p = \inf_{\mu_0 \in \mathcal{M}_d^0(B)} \|\mu_0\|_d^0 + \int p(x) |\mu(de) - \mu_0(de)| \ . \tag{9}$$

is a norm on $\mathcal{M}(K)$. Moreover, condition (7) implies that it is always a good choice to transfer some mass instead of wasting it. As a consequence, we get the following lemma [17].

Lemma 2.1. *Let p be a "waste" function satisfying (7) and (8). Then for any μ in $\mathcal{M}^0(K)$, we have*

$$\|\mu\|_p = \|\mu\|_d^0 \ . \tag{10}$$

Then the K-norm is a true extension of the Kantorovich norm.

2.0.2 The H-norm

A major drawback of the K-norm is the very restrictive assumption of finite diameter for K. The extension proposed by Hanin in [12] removes this assumption. The H-norm is defined by

$$\|\mu\|_H = \inf_{\mu_0 \in \mathcal{M}_d^0(B)} \left[\|\mu_0\|_d^0 + Var(\mu - \mu_0)\right] \ . \tag{11}$$

Though carrying good properties in relationship with the space of Lipschitz functions (see [12],[13]), it is possible to see that this norm differs from the former one on the subspace $\mathcal{M}^0(K)$. Indeed, taking $K = \mathbb{R}^d$ and $\mu = \delta_x - \delta_y$ with $|x - y| > 2$, we see that

$$\|\mu\|_0 = |x - y| \ ,$$
$$\|\mu\|_H = 2 \ .$$

In fact, the K-norm and the H-norm are conceptually very close. Indeed, ignoring (7) and taking $p \equiv 1$, the H-norm is recovered from the K-norm. It seems then natural to relax this constraint for some less restrictive one

$$\exists \alpha > 0, \ s.t. \ \forall x \in B, \ p(x) \geq \alpha \ . \tag{12}$$

We see then that formulation (9) is some kind of "unified" framework. Following [13], it is possible to prove the following theorem

Theorem 2.2. *Assume that p satisfies (8) and (12). Then $\|..\|_p$ is a norm on $\mathcal{M}(K)$.*

This H-norm differs from the Kantorovich norm on the set $\mathcal{M}_0(K)$, but stay close, in particular when the diameter of the support of μ is small. This result in stated in the following lemma [13].

Lemma 2.3. *Assume that p is a constant function. Then the norm defined in (9) satisfies the following inequalities.*

$$\|\mu\|_p \leq \|\mu\|_d^0 \leq \frac{1}{2p} \max\{diam(supp(\mu)), 2p\} \|\mu\|_p, \ \ \mu \in \mathcal{M}_d^0(K) \ . \tag{13}$$

Proof. Since the proof is not detailed in [13], we provide some explanations. Taking $\nu = \mu$ in (11), we get the first part of the inequality. Using the triangle inequality and lemma 1.2, we have that for any ν in $\mathcal{M}_0(K)$

$$\begin{aligned}
\|\mu\|_d^0 &\leq \|\mu - \nu\|_d^0 + \|\nu\|_d^0 \ , \\
&\leq \|\nu\|_d^0 + \frac{1}{2} diam(supp(\mu - \nu)) Var(\mu - \nu) \ , \\
&\leq \frac{1}{2p} \max\{2p, diam(supp(\mu - \nu))\} \left[\|\nu\|_d^0 + pVar(\mu - \nu)\right] \ .
\end{aligned}$$

If μ has a finite support, the optimal measure ν can be computed explicitly (see next section), and it is easy to show that $\nu_+ \leq \mu_+$ and $\nu_- \leq \mu_-$. This implies that $supp(\nu) \subset supp(\mu)$, which in particular means that $diam(supp(\mu - \nu)) \leq diam(supp(\mu))$. Then lemma is proved for finite measures. Finally, the proof of the whole lemma can be achieved using the density lemmas of the second section in [13]. □

This lemma shows that this choice of extension is relevant with respect to the original Kantorovich norm. Indeed, if μ is a measure of $\mathcal{M}_0$ such that $diam(supp(\mu)) \leq 2p$, then $\|\mu\|_p = \|\mu\|_d^0$. Moreover, one of the main interests in those extensions is to extend the fact that the Kantorovich norm (and the Wasserstein metric in a slightly different setting) metrizes the weak$*$ convergence of measures with zero total mass. Indeed, this is the purpose of the following theorem.

Theorem 2.4. *Suppose that (K,d) is a separable metric space. Let (μ_n) be a sequence of measures in the set $B(R) = \{\mu \in \mathcal{M}(K)$ such that $Var(\mu) \leq R\}$, where $R \in \mathbb{R}^+$. Let μ be a measure in $\mathcal{M}(K)$. Then μ_n converges weakly* to μ if and only if $\|\mu_n - \mu\|_d$ goes to 0 as n goes to ∞.*

This theorem is stated and proved in [13]. It is especially attractive from a computational point of view, since it shows that the extended Kantorovich norms are much more sensitive to oscillations than the total variation.

Remark 2.5. *Hanin considers only the case $p \equiv 1$, which seems to be the better choice from a functional analysis point of view. However, our aim is here to use the extended Kantorovich norms as an efficient tool (at least) in the field of signal processing. As we will see, it may be of some interest to consider p as a varying parameter, giving more or less local informations.*

Economical interpretation

Let e_1 and e_2 be two borel sets. As for the Kantorovich norm, the value $\psi(e_1 \times e_2)$ can be interpreted as the amount of goods carried from the suppliers located in e_1 to the demanders located in e_2. However, the assumption $\psi(e \times K) = \mu_-(e)$ is relaxed, and the difference $\mu_-(e) - \psi(e \times K)$ can be seen as the amount of demand which is not satisfied. But here we remind that the waste function p penalizes this lack of satisfaction. We have two intuitive explanations. p can be seen as

- a social cost for dissatisfaction,
- the cost required to import goods.

In the same way, the assumption $\psi(K \times e) = \mu_+(e)$ is relaxed, and the difference $\mu_+(e) - \psi(K \times e)$ can be seen as the amount of supply which is not going to be sold. Once again, we have two competitive explanations on the economic meaning of the "waste function" p. Now p can be seen as

- the storing cost,
- the cost required to export goods.

Following this interpretation, we see that it is sometimes better for the supplier to store his goods than to satisfy some distant needs. However, if the waste function is huge, as in the case of the K-norm, demands would be as much satisfied as possible.

Now we end with this presentation of extended Kantorovich norms. We shall notice that some other works can lead to adaptations of mass transport techniques even for unbalanced problems, where the measures are only required to be non-negative. We refer in particular to [5], where a numerical method is provided.

3 The discrete distance

Since our aim is to take advantage of transportation like metrics in the field of optimization, it is of primary importance to write a discrete analogous of the K-norm. Let $X \in \mathbb{R}^d$. We want to show that the evaluation of the K-norm of X can be written as a transportation cost. Let $S = X_+$ and $D = X_-$ (hence $X = S - D$). Assuming that X has a null mean, and letting $C = (c_{ij})$ be a cost matrix, it is possible to compute the transport cost from S to D. This cost, denoted by $\|S - D\|$, defines the discrete Kantorovich metric between S and D. As in the previous section, we want to extend this metric over the whole space $\mathbb{R}^d$. We therefore introduce the waste function $P = (p_i)$. For any X in $\mathbb{R}^d$, the K-norm of X is defined by

$$\|X\|_p = \inf_{X_0 \in \mathbb{R}_0^d} \|X_0\|_d^0 + \sum_{i=1}^{d} p_i |x_i^0 - x_i| \,. \tag{14}$$

This formula leads to the consideration of a new transportation problem, which can be thought as an unbalanced mass transportation problem (UMTP) : given $S \in \mathbb{R}^n$ and $D \in \mathbb{R}^m$, given a cost matrix $C \in M_{nm}(\mathbb{R})$ and two waste vectors $P \in \mathbb{R}^n$ and $Q \in \mathbb{R}^m$, minimize

$$E_{cp}(Y) = \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}y_{ij} + \sum_{i=1}^{n} p_i(s_i - \sum_{j=1}^{m} y_{ij}) + \sum_{j=1}^{m} q_j(d_j - \sum_{i=1}^{n} y_{ij}) \ , \tag{15}$$

in the set of matrix $Y = (y_{ij})$ such that

$$\forall (i,j) \in [1;n] \times [1;m], \ y_{ij} \geq 0 \ , \tag{16}$$

$$\forall i \in [1;n], \ \sum_{j=1}^{m} y_{ij} \leq s_i \ , \tag{17}$$

$$\forall j \in [1;m], \ \sum_{i=1}^{n} y_{ij} \leq d_j \ . \tag{18}$$

The UMTP can then be seen as a transportation problem with (partially) relaxed marginals constraints. It differs however from the one considered in [23]. In order to compute the optimal cost of this transport, it is of great interest to reduce the UMTP to the classical mass transport problem. It can be done by adding some artificial mass. Indeed, an expansion of (15) gives the following lemma.

Lemma 3.1. *Let $m_0 = \sum_{i=1}^{n} s_i$ and $m_1 = \sum_{j=1}^{m} d_j$. Let $\tilde{S} \in \mathbb{R}^{n+1}$ and $\tilde{D} \in \mathbb{R}^{m+1}$ such that*

$$\tilde{s}_i = \begin{cases} s_i & \text{if } \ i \in [1;n] \ , \\ m_1 & \text{if } \ i = n+1 \ , \end{cases} \quad \text{and} \quad \tilde{d}_j = \begin{cases} d_j & \text{if } \ j \in [1;m] \ , \\ m_0 & \text{if } \ j = m+1 \ . \end{cases}$$

Let $\tilde{C}$ be the extended cost matrix defined by

$$\tilde{C} = \begin{pmatrix} c_{11} & \dots & c_{1m} & p_1 \\ \dots & \ddots & \vdots & \vdots \\ c_{n1} & \dots & c_{nm} & p_n \\ q_1 & \dots & q_m & 0 \end{pmatrix} \tag{19}$$

Then the cost of the optimal transport from $\tilde{S}$ to $\tilde{D}$ is equal to the optimal cost of the UMTP.

This reduction allows to use the numerous results on optimal transportation. In particular, the simplex method runs in polynomial time. But since we want to use extended Kantorovich norms in practical applications, we are dependent on the computational cost of this norm, which is very expensive in the general case. Indeed, assume that for all (i,j), we have $s_i = d_j = 1$. Then the efficient primal-dual simplex method developed in [2] runs in a average of $O((n+m)^2 \log(n+m))$ operations, whereas the total variation between S and D is computed in $O(n+m)$ operations. Hence the computation cost is much too expensive, and has to be improved for practical applications.

4 Model simplifications : Monge matrices

It appears that we have to seek for simplifications if we want to make a practical use of extended Kantorovich norms. It is well-known that the optimal mass transport problem can be solved very efficiently when the cost function satisfies the Monge property (or quadrangle inequality)

$$\begin{gathered} \forall (i,j), \ 1 \leq i \leq n-1, \ 1 \leq j \leq m-1, \\ c[i,j] + c[i+1,j+1] \leq c[i+1,j] + c[i,j+1] \ . \end{gathered} \tag{20}$$

In this case, the North-West corner rule solves the allocation problem in $O(n+m)$ operations (see [14]). It is therefore natural to expect some improvement on the speed of Balinski's method when the cost matrix C is a Monge matrix. Moreover, this assumption is always satisfied in the case of extended Kantorovich norms when $K = \mathbb{R}$. This special structure arises in much more transportation problems, as remarked below.

Remark 4.1. *Many (one-dimensional) transportation problems lead to bitonic Monge matrices. Indeed, in the special case of the extended Kantorovich norm, C is the distance matrix between two ordered sets of points on the line, (x_i) and (y_i). In this case, it is easy to see that the quadrangle inequality is satisfied. In a more general setting, consider $c(i,j) = \|x_i - y_j\|^p$, with $p > 1$, and still assume that the two sets of points are located on the same line. In this case, the quadrangle inequality is a straightforward consequence of the convexity of $\|.\|^p$.*

However, the reduction to an allocation problem (as described in lemma 3.1) breaks down this particular structure, and a simple application of the North-West corner rule will not anymore lead to the solution. Nevertheless, it is possible to dream of some near optimal solution. Indeed, once the set of exported supplies and imported demands are known, the problem is reduced to the optimal mass transport problem with a cost matrix satisfying the quadrangle inequality. It means that the solution of the UMTP can be written in an import vector of size n, an export vector of size m, and a path in the matrix, whose size is $O(n+m)$. Then the solution of the UMTP consists in $O(n+m)$ terms, and it is still possible to expect a significant improvement compared to general methods. An other natural simplification deals with the waste vectors P and Q. Following Hanin, a good choice seems to take them constant, and it is always possible to reduce the problem to an equivalent problem where they are indeed constant.

Lemma 4.2. *Let $\bar{P}$ and $\bar{Q}$ be two vectors of size $n+1$ and $m+1$. Then the solution Y of the UMTP is not changed if the cost matrix $\tilde{C}$ is replaced by a matrix $\bar{C} = (\bar{c}_{ij})$ defined as follows.*

$$\forall (i,j) \in [1; n+1] \times [1; m+1],\ \bar{c}_{ij} = \tilde{c}_{ij} + \bar{p}_i + \bar{q}_j \ . \tag{21}$$

For example, let $k \in \mathbb{R}^+$, and extend the waste vectors P and Q with $p_{n+1} = q_{m+1} = 0$. Define the special matrix $\bar{C}$ as follows.

$$\forall (i,j) \in [1; n+1] \times [1; m+1],\ \bar{c}_{ij} = \tilde{c}_{ij} + k - p_i - q_j \ .$$

Then the $(n+1)^{th}$ line and the $(m+1)^{th}$ column of $\bar{C}$ are constant and "equal" to k (except for their last element, which is 0).

We have to notice that such a perturbation of the problem preserves Monge property. Therefore, we can always assume the "waste" vectors to be constant (we speak then of a "waste parameter" p). To the best of our knowledge, only few results deal with unbalanced mass transportation problems, especially when the cost matrix satisfies Monge property. Indeed, this problem is only briefly mentioned in the review paper of Burkard and al. [6]. In the case of integral data, a first result has been obtained by Karp and Li in [20], where the cost matrix describes the distance between two sets of points (x_i) and (y_j) on the line. This result has been improved in [1], where the perfect matching problem is considered. This problem (which we will denote by the perfect wedding problem throughout this section) can be stated as follows. Given two sets $[1; N]$ and $[1; M]$ of sinks and sources (with $N \leq M$) and a $N \times M$ cost matrix C, find an injective function φ which minimizes the cost

$$S_0 = \sum_{i=1}^{N} c[i, \varphi(i)] \ . \tag{22}$$

This problem can be seen as a perfect wedding problem. Indeed, given a set of men $[1; N]$ (blue points) and a set of women $[1; M]$ (pink points), we have to find the set of couples which maximizes the amount of joy (or, since wedding is for the best and the worst, which minimizes the amount of pain). This perfect wedding problem is closely linked to our problem (assuming integer supplies and demands) : contrary to this problem, we allow single-persons, even when there are both blue and pink single-points. However, the price to pay for choosing loneliness is the disapproving glance of other people, described by our "waste" function p. From now on in this section, p would be a constant function (a parameter), and would be thought as a social pressure. Then it is easy to see that when this social pressure is very high, loneliness is never chosen in an optimal allocation. Therefore, our problem reduces to the perfect wedding problem in this special case. In the sequel, we look at a sequence of problems, the n^{th} single-person problem, which are something like a link between our wedding problem and the perfect wedding problem.

4.1 The n^{th} single-person problem

Let C be a $N \times M$ cost matrix. The n^{th}-single person problem $(\mathcal{P}_n)$ is to find the set I_n and J_n such that

$$Card(I_n) = n \; , \tag{23}$$

$$Card(J_n) = M - N + n \; , \tag{24}$$

and such that the assignment problem from the set $[1; N]\backslash I_n$ to the set $[1; M]\backslash J_n$ has the minimal optimal cost. The optimal allocation is then defined by

$$S_n = \sum_{i=1}^{N-n} c[\varphi(i), \psi(i)] \; , \tag{25}$$

where φ and ψ are two injective functions such that

$$\varphi : [1; N-n] \rightarrow [1; N]\backslash I_n \; , \tag{26}$$

$$\psi : [1; N-n] \rightarrow [1; M]\backslash J_n \; . \tag{27}$$

In order to avoid permutations, we can assume that φ is increasing (and then uniquely determined once I_n is chosen).

4.1.1 Connection with the wedding problem

It is obvious that the 0^{th} single-person problem is the perfect wedding problem. The intuitive connection between the wedding problem and the n^{th} single-person problems is stated in the following lemma.

Lemma 4.3. *Let (N, M) in $\mathbb{N}^2$ such that $N \leq M$. Let C be a $N \times M$ cost matrix. Then for each waste parameter p in $\mathbb{R}^+$, there exists some n_p in $\mathbb{N}$ such that $E_p = S_{n_p} + (2n_p + M - N)p$. Moreover, n_p is a decreasing function of p.*

Proof. The first part of the lemma is obvious, and we focus on the second part. Let $0 < p_0 < p_1$ be two real waste parameters, and write $n_{p_i} = n_i$. We have

$$S_{n_0} + (2n_0 + M - N)p_0 \leq S_{n_1} + (2n_1 + M - N)p_0 \; , \tag{28}$$

$$S_{n_1} + (2n_1 + M - N)p_1 \leq S_{n_0} + (2n_0 + M - N)p_1 \; . \tag{29}$$

We get then

$$S_{n_1} - S_{n_0} \leq p_1(n_0 - n_1) \; , \tag{30}$$

$$S_{n_0} - S_{n_1} \leq p_0(n_1 - n_0) \; . \tag{31}$$

We deduce that $p_0(n_0 - n_1) \leq p_1(n_0 - n_1)$, which means that $n_0 \geq n_1$. Hence n_p is a decreasing function of p. This achieves the proof of lemma 4.3. $\square$

We have seen that the number of single-points increases as the social pressure decreases. A natural question is to ask whether single-persons can take advantage of the rise of the social pressure to find one sweet heart. If not, we say that the cost matrix satisfies the "greedy property".

Definition 4.4. A greedy property. *Let (N, M) in $\mathbb{N}^2$ such that $N \leq M$. Let C be a $N \times M$ cost matrix. Then C is said to satisfy the greedy property if for any n in $[0; N-1]$, we have*

$$\begin{cases} I_n \subset I_{n+1} \\ J_n \subset J_{n+1} \end{cases} \tag{32}$$

As we will see in the following example, the "greedy property" is not satisfied for general cost matrices.

4.1.2 A counter example

Consider the n^{th} single-person problem with the following cost matrix.

$$C = \begin{pmatrix} 2 & 5 & 1 & 5 & 5 \\ 5 & 2 & 5 & 1 & 5 \\ 5 & 5 & 2 & 5 & 5 \\ 5 & 5 & 5 & 2 & 5 \\ 5 & 5 & 5 & 5 & 3 \end{pmatrix}$$

The solutions of the problem are shown in the table below.

n	I_n	J_n
0	$\emptyset$	$\emptyset$
1	$\{5\}$	$\{5\}$
2	$\{3, 4\}$	$\{1, 2\}$

We see that the "greedy property" is not satisfied here.

4.1.3 Monge matrices and the n^{th} single-person problem

The "greedy property" is not satisfied for a general cost matrix. However, we can expect that some special structured matrices always satisfy it. Indeed, it is the case when the cost matrix satisfies the Monge property, as will be proved in theorem 4.6. One can ask whether the solution is even much simpler in this special case, meaning that the growing sequence of single-persons is the same once the matrix is known to be a Monge matrix. It is false, and any fixed set $I_n \times J_n$ can be the unique minimizer for the n^{th} single-person problem $\mathcal{P}_n$ with an appropriate cost matrix. The construction of such a matrix is indicated in the following lemma, which is stated before theorem 4.6 because it allows to use an unicity assumption in its proof.

Lemma 4.5. *Let $(N, M) \in \mathbb{N}^2$. Assume that $N \leq M$ and fix n in $[0; N]$. Let I_n and J_n be two subsets of $[0; N]$ and $[0; M]$ satisfying (23) and (24). Then there exists a $N \times M$ Monge matrix C such that $I_n \times J_n$ is the unique solution of $\mathcal{P}_n$.*

Proof. We describe here how to build such a matrix. Let $X = (x_i)_{i=1..N}$ and $Y = (y_i)_{i=1..M}$. Set $x_i = i$. From the Monge property, we know that the optimal cost is given by

$$S_n = \sum_{i=1}^{N-n} c[\varphi(i), \psi(i)] \ , \tag{33}$$

where φ and ψ are two strictly increasing functions such that

$$\varphi : [1; N-n] \to [1; N] ,$$
$$\psi : [1; N-n] \to [1; M] .$$

For $i \in [0; N-n]$, we set $y_{\psi(i)} = x_{\varphi(i)}$. The other y_i are defined such that the sequence (y_i) is strictly increasing and that y_i is an integer if and only if i is in the range of ψ. Finally, the cost function c is given by $c[i,j] = d(x_i, y_j)$. It is easy to see that C is a Monge matrix and that $I_n \times J_n$ is the unique solution of $\mathcal{P}_n$. This achieves the proof of the lemma 4.5. $\square$

Theorem 4.6. *Let C be a $N \times M$ Monge matrix. Let $n \in [0; M-N-1]$, and denote by $I_n \times J_n$ a solution of $\mathcal{P}_n$. Then there exists a solution $I_{n+1} \times J_{n+1}$ of $\mathcal{P}_{n+1}$ such that*

$$\begin{cases} I_n \subset I_{n+1} , \\ J_n \subset J_{n+1} . \end{cases}$$

Proof. Up to a small perturbation of the cost matrix C (see lemma 4.5), preserving Monge property, we can assume that $I_n \times J_n$ is the unique solution of $\mathcal{P}_n$. Denote by $m = M - N - n$ the number of marriages for $\mathcal{P}_n$. Those pairs are denoted by $p_n(k) = (i_n(k), j_n(k))$, where k belongs to $[1; m]$. The $(m-1)$ pairs p_{n+1} are defined in the same way. Set $\mathcal{E}_n = \{p_n(k)\} \cup \{p_{n+1}(k)\}$, and denote by $\tilde{p}(k) = (\tilde{i}(k), \tilde{j}(k))$ the elements of $\mathcal{E}_n$. Define $S = E_n + E_{n+1}$. Precisely, we have

$$S = \sum_{k=1}^{2m-1} c[\tilde{p}(k)] . \tag{34}$$

Since C is a Monge matrix, we see that any pair of marriages $(\tilde{p}(k), \tilde{p}(k'))$ such that $\tilde{i}(k) < \tilde{i}(k')$ and $\tilde{j}(k') < \tilde{j}(k)$ can be rearranged in order to decrease the cost. Indeed, the quadrangle inequality applied to $\tilde{i}(k)$, $\tilde{i}(k')$, $\tilde{j}(k')$ and $\tilde{j}(k)$ is simply

$$c[\tilde{i}(k), \tilde{j}(k')] + c[\tilde{i}(k'), \tilde{j}(k)] \le c[\tilde{i}(k), \tilde{j}(k)] + c[\tilde{i}(k'), \tilde{j}(k')] .$$

We deduce that it is possible to get a new set $\mathcal{F} = \{\bar{p}(k),\ k \in [0, 2m-1]\}$ satisfying the following properties

$$Card(\mathcal{F}) = 2m - 1 , \tag{35}$$

$$\forall (k, k'),\ (\bar{i}_{k'} > \bar{i}_k) \implies (\bar{j}_{k'} \ge \bar{j}_k) , \tag{36}$$

$$\forall i_0 \in [0; N],\ Card(\{k \in [0; 2m-1]\ s.t.\ \bar{i}(k) = i_0\}) \le 2 , \tag{37}$$

$$\forall j_0 \in [0; M],\ Card(\{k \in [0; 2m-1]\ s.t.\ \bar{j}(k) = j_0\}) \le 2 , \tag{38}$$

$$\sum_{k=1}^{2m-1} c[\bar{p}(k)] \le S . \tag{39}$$

Introduce the order $\ll$ on the locations on the allocation plan. Let $p_0 = (i_0, j_0)$ and $p_1 = (i_1, j_1)$ be two locations. Then

$$p_0 \ll p_1 \iff d(p_0) \le d(p_1) . \tag{40}$$

Now we order the set $\mathcal{F}$ so that $d(\bar{p}(.))$ is non-decreasing. From the above properties, we have that

$$\forall 1 \le k \le k' \le 2m-1,\ (\bar{i}(k) < \bar{i}(k')) \implies (\bar{j}(k) \le \bar{j}(k')) , \tag{41}$$

$$\forall 1 \le k \le k' \le 2m-1,\ d(\bar{p}(k)) = d(\bar{p}(k')) \implies (p(k) = p(k')) . \tag{42}$$

We are going now to build two sets $\mathcal{K}_n$ and $\mathcal{K}_{n+1}$ defining optimal allocations for $\mathcal{P}_n$ and $\mathcal{P}_{n+1}$. Some points can be twice in $\mathcal{F}$. Then remove both of them from $\mathcal{F}$ and send them respectively into $\mathcal{K}_n$ and $\mathcal{K}_{n+1}$. We deduce from (41) that $\bar{d}$ is now strictly increasing on $\mathcal{F}$. We have $\mathcal{F} = \{f_1, f_2, ..., f_{2m'+1}\}$. Notice that at most two points of $\mathcal{F}$ may have the same abscissa, and that those points have then to follow each other. Then we send $\{f_{2k+1}\}$ into $\mathcal{K}_n$ and $\{f_{2k}\}$ into $\mathcal{K}_{n+1}$. We have then

$$Card(\mathcal{K}_n) = m \ , \tag{43}$$
$$Card(\mathcal{K}_{n+1}) = m - 1 \ . \tag{44}$$

>From the above observation, we see that the allocation plan defined by $\mathcal{K}_n$ (resp. $\mathcal{K}_{n+1}$) is admissible for $\mathcal{P}_n$ (resp. $\mathcal{P}_{n+1}$). Denote by $\tilde{E}_n$ and $\tilde{E}_{n+1}$ the related costs. We get

$$\tilde{E}_n + \tilde{E}_{n+1} \leq E_n + E_{n+1} \ . \tag{45}$$

We deduce from this inequality that those allocation plans are optimal. >From the unicity of the solution for $\mathcal{P}_n$, we conclude that the set $\tilde{I}_n$ defined with $\mathcal{K}_n$ is in fact I_n. It remains to show that $\tilde{I}_n \subset \tilde{I}_{n+1}$. Without loss of generality, we can assume that

$$\tilde{I}_n \cap \tilde{I}_{n+1} = \emptyset \ , \tag{46}$$
$$\tilde{J}_n \cap \tilde{J}_{n+1} = \emptyset \ , \tag{47}$$
$$\mathcal{K}_n \cap \mathcal{K}_{n+1} = \emptyset \ . \tag{48}$$

>From now on, we drop the tildes. We have to mention that m and n have changed within this restriction of the problem. By construction, two points of $\mathcal{K}_n$ are separated by a single point of $\mathcal{K}_{n+1}$ (and conversely). Assume that two consecutive (distinct) points f_{k_0} and f_{k_0+1} of $\mathcal{F}$ satisfy $d(f_{k_0+1}) \geq d(f_{k_0}) + 2$. Then (46) and (47) imply that

$$i(k_0) < i(k_0 + 1) \ , \tag{49}$$
$$j(k_0) < j(k_0 + 1) \ . \tag{50}$$

In particular, it means that it is possible to split the allocation plans into four parts. Indeed any points of $\mathcal{F}$ have to be located in one of the areas A and B, where the lines separate f_{k_0} and f_{k_0+1}.

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) \tag{51}$$

For parity reasons, one of the locations contains the same number of points of $\mathcal{K}_n$ and $\mathcal{K}_{n+1}$. Then on this area, say A, those restricted allocations minimize the cost for some n^{th} single-person problem. From the unicity assumption on I_n and J_n, we deduce that those allocations are identical on A. This contradicts (48). As a consequence, we get

$$\forall k \in [0; m], \ d(p_n(k)) = 2k - 1 \ , \tag{52}$$
$$\forall k \in [0; m-1], \ d(p_{n+1}(k)) = 2k \ . \tag{53}$$

Up to a transposition, we can assume that $i_{n+1}(1) = i_n(1) + 1$. This case is illustrated on the following 3×3 matrix, where the squares stand for the locations of points of $\mathcal{K}_n$ (in black) and $\mathcal{K}_{n+1}$ (in white).

$$\left(\begin{array}{c|c|c} \blacksquare & \square & \\ \hline & \blacksquare & \square \\ \hline & & \blacksquare \end{array}\right)$$

For this problem, we have $I_n = \emptyset$, $J_n = \emptyset$ and $I_{n+1} = \{m\}$ and $I_{n+1} = \{1\}$. The symmetric (transposed) case, for which $j_{n+1}(1) = j_n(1) + 1$ leads to $I_n = \emptyset$, $J_n = \emptyset$ and $I_{n+1} = \{1\}$ and $I_{n+1} = \{m\}$. Coming back to our original problem, we get that

$$I_n \backslash (I_n \cap I_{n+1}) = \emptyset \ , \tag{54}$$
$$J_n \backslash (I_n \cap J_{n+1}) = \emptyset \ , \tag{55}$$

which precisely means that $I_n \subset I_{n+1}$ and $J_n \subset J_{n+1}$. This achieves the proof of theorem 4.6. $\square$

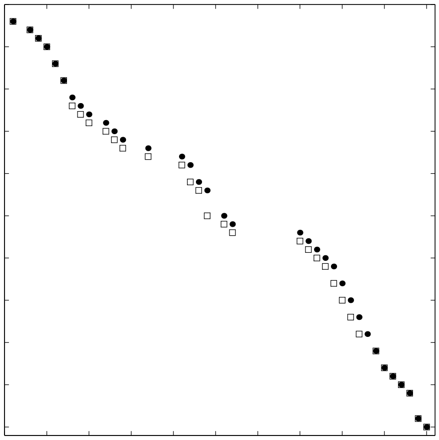

Figure 1: South-West shift

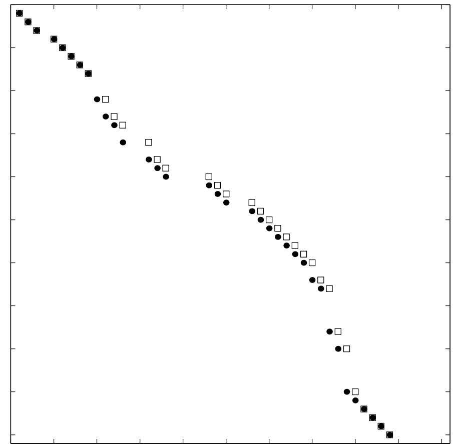

Figure 2: North-East shift

Once the "greedy property" is known to hold, it is interesting to wonder whether there exists a criterion allowing to check that the final number of single-persons (for a fixed waste parameter p) is reached. The following theorem provides such a criterion : the final set of single-points is reached as soon as any new pair of single-points increases the cost.

Theorem 4.7. *Let C be a $N \times N$ Monge matrix. Define*

$$\forall k \in [1;N],\ a_k = S_{k-1} - S_k \ . \tag{56}$$

Then the sequence $(a_k)_{k\in[1;N]}$ is non-increasing.

Proof. In order to prove theorem 4.7, it is sufficient to show that $\delta_1 \geq \delta_2$. In the proof, we considered several cases, depending on the respective locations of the new pairs of single-persons. However, all inequalities are shown using the "optimality" of $C_1 = (i_1, j_1)$ against (i_1, j_2) and (i_2, j_1), and the quadrangle inequality when $(i_2 - i_1).(j_2 - j_1) > 0$. Not being of primary interest, details of the proof are postponed until appendix A. □

The main interest of theorems 4.6 and 4.7 is an efficient algorithm, which will be explained in the next section. However, they also allow to get the concavity of the K-norms with respect to the waste-parameter. This property is proved in appendix C.

4.2 A greedy algorithm to solve the wedding problem

As a consequence of theorem 4.6, we get a greedy algorithm to define the sequence of single-persons when C is a Monge matrix. Indeed, once I_n and J_n are known, it remains to find the next single-points to obtain I_{n+1} and J_{n+1}. At each step, the optimal allocation (of cost S_n) is simply computed using the North-West Corner Rule. As we said, the allocation plan can be viewed as a "path in the matrix" (or a generalized diagonal [1]). The next path is then simply deduced from the former one through a "shift" of some of its parts (see figure 1 and 2, where old paths are designed with points, while new ones are designed with squares). It is possible to take advantage of this simple structure to compute the corresponding new pair of single-points.

When the cost matrix is a $N \times M$ cost-matrix, we denote by the "initialization step" the reduction to a square $(N \times N)$ matrix, i.e. the computations of the first $M - N$ single-persons. Theorems 4.6 and 4.7 ensure that the following algorithm computes the solution of the wedding problem and the solution of the N^{th} single-person problem (if there is no waste function p, and then no stopping step 3).

- Step 1 : *Initialization* (reduction to a square matrix).
- Step 2 : *Computation of a new pair of single-points.*
- Step 3 : *Computation of the induced cost.*
  If this cost is non-negative, then stop.
  If not, go back to step 2.

4.2.1 Initialization

When the cost matrix is not a square matrix, we have to compute the first $M-N$ single-points. An easy way is to build a big $M \times M$ Monge matrix $\tilde{C}$ such that the restricted $N \times M$ matrix is C, and such that the first points in $\tilde{J}_M$ are $\{N+1, N+2, ..., M\}$. This can be easily done by computing some prohibitive constant K_C (e.g. $\max C[i,j]$), assigning it to $\tilde{C}[N+1, M]$ and then constructing recursively the $\tilde{C}[N+1, M-k]$ with the following requirement

$$\tilde{C}[N+1, M-k] \geq \max\left\{K_C, \tilde{C}[N, M-k] + \tilde{C}[N+1, M+1-k] - \tilde{C}[N, M+1-k]\right\}. \quad (57)$$

Remaining lines can be taken equal to this $N+1^{\text{th}}$ line. Hence we get a $M \times M$ Monge matrix, which can be used as entry in our "square algorithm". However, we see that this initialization might be very expensive if we have $M \gg N$. At this time, it is relevant to observe that coefficients in $[N+1; M] \times [1; N]$ will never be used in the computations. Hence it is possible to choose $\tilde{C}[N+1, N+1]$ huge enough to ensure that the first single-point $\tilde{j}_1$ in $\tilde{J}_M$ has to be $N+1$. As a consequence, we get that the first single-point $\tilde{i}_1$ in $\tilde{I}_M$ has to satisfy $i_1 \leq N+1$. Hence we see that it is possible to compute the first single-point in $\tilde{I}_M$ using only a $(N+1) \times (N+1)$ cost matrix. Moreover, coefficients in the $N+1^{\text{th}}$ line wouldn't be used (since $\tilde{j}_1 = N+1$), and do not even deserve to be computed. Using the method described in the next subsection, we see that this single-point can be computed in $O(N)$ operations. As a consequence, we get that the first $M-N$ single points in I_M can be computed in $O(N(M-N))$ operations.

In the special case when the cost matrix is a bitonic Monge matrix (see definition 4.8), an algorithm of Aggarwal and al. [1], based on a "divide and conquer" strategy, allow to perform the initialization step in $O(N \log M)$ operations.

Definition 4.8. *An $N \times M$ matrix C is (row) bitonic Monge if C is a Monge matrix and each row of C is bitonic. A row i_0 is bitonic if the sequence $c(i_0, .)$ is non-increasing until the minimal entry $c(i_0, j_0)$ is reached, and then non-decreasing.*

4.2.2 Computation of the next single-points

There are $(N-n)^2$ choices for the pair of divorces required to find the new single-points, but the structure of the related problem allows to perform this step much faster. Indeed, we describe here how to compute those single-points in $O(N-n)$ operations. We have $I_{n+1} = I_n \cup \{i_n(k_0)\}$ and $J_{n+1} = J_n \cup \{j_n(k_1)\}$ with $(k_0, k_1) \in [1; N-n]^2$. W.l.o.g. assume that $k_0 \leq k_1$. In this case, the cost of the two divorces is

$$A(k_0, k_1) = -c[i(k_1), j(k_1)] + \sum_{k=k_0}^{k_1-1} \left(c[i(k+1), j(k)] - c[i(k), j(k)]\right). \quad (58)$$

We have to minimize $A(k_0, k_1)$ under the constraint $k_0 \leq k_1$. We define then

$$u_{k_1} = -c[i(k_1), j(k_1)] + \sum_{k=1}^{k_1-1} \left(c[i(k+1), j(k)] - c[i(k), j(k)]\right), \quad (59)$$

$$v_{k_0} = \sum_{k=1}^{k_0-1} \left(c[i(k+1), j(k)] - c[i(k), j(k)]\right). \quad (60)$$

By construction, we have $A(k_0, k_1) = u_{k_1} - v_{k_0}$. For $k \in [1; N-n]$, define

$$U(k) = \inf_{i \geq k}\{u_i\} , \tag{61}$$

$$V(k) = \sup_{i \leq k}\{v_i\} . \tag{62}$$

$$F(k) = U(k) - V(k) , \tag{63}$$

Any value of F is the cost of an admissible divorce. Moreover, if the optimal break-up is obtained for (k_0, k_1), then for any $k_0 \leq \tilde{n} \leq k_1$, we have $F(\tilde{n}) = A(k_0, k_1)$. Hence we get an algorithm to find the optimal pair. First, find a minimizer $\tilde{n}$ of F. Second, find a minimizer k_1 of u on $[\tilde{n}; N-n]$ and a maximizer k_0 of v on $[1; \tilde{n}]$. Then (k_0, k_1) is an optimal pair. Each step of the algorithm can be performed in $O(N-n)$ operations, and we deduce that it takes the same work to compute I_{n+1} and J_{n+1} once I_n and J_n are known.

Assuming that $N = M$ (or that the $M - N$ first single-persons are given), we conclude that the global algorithm runs in $O(N^2)$ operations. For a $N \times M$ cost matrix, the cost of the initialization step has to be added. The cost of the computation of the global sets of single-persons are summarized in the following theorem.

Theorem 4.9. *Let C be an $N \times M$ Monge matrix. Then the sequence of n^{th} single-persons problem $(\mathcal{P}_n)$ can be solved in*

- *$O(NM)$ operations in the "general" case.*
- *$O(N \log M + N^2)$ operations if C is bitonic.*

As a consequence of theorem 4.9, we get that in both cases, our algorithm solves the wedding problem much faster than Balinski's method.

4.3 Numerical results

Classical methods for solving the Linear Assignment Problem (on square matrices) run in $O(n^3)$ operations. Amongst them, the algorithm LAPJV [15] of Jonker and Volgenant seems to be efficient and is still used in recent applications [3]. This algorithm will be used as a reference in order to evaluate the practical behavior of our method. In table 1, our algorithm is denoted by GASPP (greedy algorithm for the single-person problem). In the test, we did not solve the wedding problem, but the single-person problem, which is actually solved in the "worst case" of the wedding problem (when the waste parameter is very small, there are no marriages). However, the accuracy of our method was checked for each random problem, and no error occurred.

Both algorithms have been tested on 200 randomly generated Monge matrices. The computational results are for Fortran 77 codes on a Compaq PWS500 with a 500 MHz processor α 21164. Average running times are presented.

Problem size	LAPJV	GASPP	GASPP/LAPJV
$n = 50$	1.49	0.37	0.25
$n = 100$	7.38	1.24	0.17
$n = 250$	59.6	8.45	0.15
$n = 500$	352.5	37.3	0.11

Table 1: Computation times for Monge matrices (in ms)

While theoretical complexities indicate that our algorithm is well suited for large data, numerical results show that GASPP run faster than LAPJV even for small problems.

4.4 Monge matrices and real-valued functions

Now we go back to the Unballanced Mass Transportation Problem defined in section 3, but the cost matrix C is still assumed to be a Monge Matrix. The algorithm GASPP can be adapted to solve this problem, even when the demand and supply vectors are real.

Using the Monge property, we know that the optimal allocation plan Y can be expressed as a weighted "path" in the matrix starting from the North-West corner and ending at the South-East corner (remember the introduction of section 4). The path can be defined through a triplet $L = (a, i, j)$ of vectors of size (at most) $n + m - 1$. The transport cost is then

$$E = \sum_{k=1}^{n+m-1} c[i(k), j(k)]a(k) + p\left(\sum_{k=1}^{n} Im(k) + \sum_{k=1}^{m} Ex(k)\right) , \tag{64}$$

where Im (resp. Ex) stands for the import (resp. export) vector.

In order to explain the method, we remind what we called a "shift" (see figures 1 and 2). Assume that the first balanced path is given (i.e. there is no more waste than required), then we have to compute some finally wasted demands and supplies. This step is very similar to the computation of a new pair on single-persons. As in this case, part of the path will be shifted either in the North-East or in the South-West. A shift is said to be optimal if the induced cost per amount of new wasted goods is the smallest. The initialization step is also very similar to the one of the algorithm for the wedding problem. The new algorithm is described below.

- Step 1 : *Initialization* (reduction to a balanced problem).
- Step 2 : *Computation of the optimal shift.*
- Step 3 : *Computation of the induced cost.*
  If this cost is non-negative, then stop.
  If not, shift until some y_{ij} vanishes. Then go back to step 2.

The use of real supply and demand vectors requires some special care when the question turns to the programming of the method. Indeed, the computation of the cost of a shift has to take into account some "zero-weight" turnings in the path. Those turnings have to be handled in different ways, according to the expected direction of the shift. This harms the efficiency of the method, and numerical computations indicate that the "real adapted code" runs about 2.5 times slower that the "integer code" for integer supply and demand vectors.

In the wedding problem, any shift reduces the size of the new problem. Hence the evaluation of the theoretical complexity is quite easy. In the real case however, several shifts may be needed for such a reduction. In the following example, a shift is presented. The related effects on the weighted path are described through arithmetical operators. The locations of the wasted goods are indicated in the same way, outside the double bars.

$$\left(\begin{array}{c||c|c|c|c|c}
 & = & & & & \\ \hline
+ & = & - & & & \\ \hline
 & & + & = & - & \\ \hline
 & & & & + & - \\ \hline
 & & & & & = \\ \hline\hline
 & & & & & +
\end{array}\right)$$

Then if the limiting weight is obtained for the second north-east corner ($y_{3,4}$), we see that the size of the resulting problem does not decrease. Hence it is interesting to evaluate the effective complexity of the method. We therefore tested the algorithm on randomly generated Monge matrices (see remark 4.10), and computed the required number of shifts to waste any demand and supply. Average and worst-case results are presented on figures 3 and 4.

The curves on figures 3 and 4 are well fitted by polynomials of order 2, so that we expect an effective running time of $O(n^3)$.

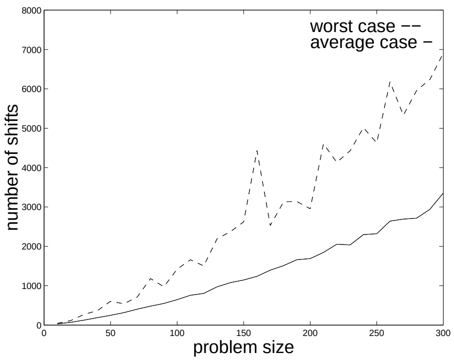

Figure 3: Number of shifts (case $p = 1$)

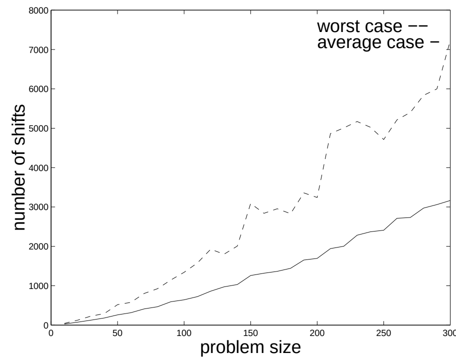

Figure 4: Number of shifts (case $p = 2$)

Remark 4.10. *Our tests are performed on randomly generated Monge matrices, following the method described in remark 4.1 (with $p = 1$ for figure 3 and $p = 2$ for figure 4). However, our process is such that particular (e.g. non-bitonic) matrices may not be obtained. As a consequence, results shall be relativized.*

5 K-Norm of a translated signal

In this section, we investigate the behavior of the K-norm on translated signals. We first deal with the second derivative of a Gaussian, which is plotted in figure 5.

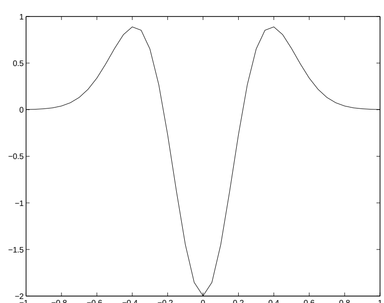

Figure 5: Second derivative of a Gaussian

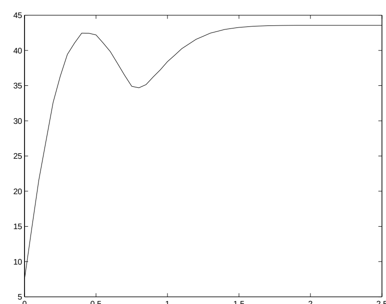

Figure 6: L^1 cost of a translation

As expected, the L^1 norm between the signal and its translations oscillates (see figure 6), and there is a local minimum. On figure 7, the K-norm is used with 20 waste parameters p varying from 0.05 to 1. Since the L^1 norm can be recovered as a limiting case of extended Kantorovich norms (see appendix B), local minima may exist for low value of the waste parameter. As p increases however, we see that the local extrema move away from the global minimum (see figure 8), and even disappear for high value of p. Hence the range where the global minimum can be successfully searched increases.

On figure 6, we see that the right derivative in 0 is very high. Indeed, the L^1 norm of a translation might even not be continuous (e.g. if the signal is a finite sum of Dirac functions). When a K-norm is used, the cost of a translation is continuous, and even Lipschitz, as shown in the following lemma.

Lemma 5.1. *Let $p \in \mathbb{R}$ and $(f, g) \in \mathcal{M}(\mathbb{R})^2$. For $\tau \in \mathbb{R}$, the function g_τ is the translation of g.*

$$g_\tau : x \mapsto g(x - \tau) \; . \tag{65}$$

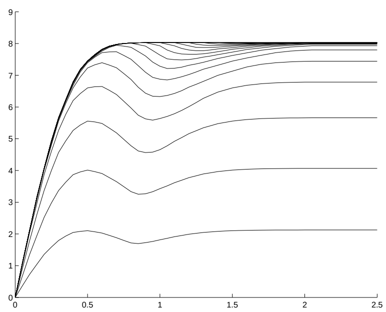

Figure 7: K-norm of a translation

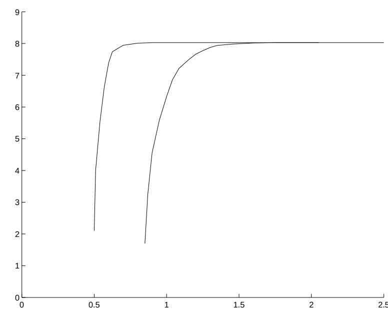

Figure 8: "Trajectories" of local extrema

Then the application ψ *defined by*

$$\psi : \tau \mapsto \|g_\tau - f\|_p \ , \tag{66}$$

is Lipschitz with constant $\min(Var(g), Var(f))$.

Proof. Noticing that for any τ in $\mathbb{R}$, $\|g_\tau - f\|_{L^1(\mathbb{R})} = \|g - f_{-\tau}\|_{L^1(\mathbb{R})}$ and using the triangle inequality, we get

$$\begin{aligned} |\psi(\tau_1) - \psi(\tau_0)| &= \|g_{\tau_1} - f\|_p - \|g_{\tau_0} - f\|_p \\ &\leq \min(\|g_{\tau_1} - g_{\tau_0}\|_p, \|f_{-\tau_1} - f_{-\tau_0}\|_p) \ , \\ &\leq \min(\|g_{(\tau_1-\tau_0)} - g\|_p, \|f_{(\tau_0-\tau_1)} - f\|_p \ , \\ &\leq \min(\|g_{(\tau_1-\tau_0)} - g\|_d^0, \|f_{(\tau_0-\tau_1)} - f\|_d^0) \ . \end{aligned}$$

Using the transport plan associated to a simple translation, we get

$$|\psi(\tau_1) - \psi(\tau_0)| \leq |\tau_1 - \tau_0| \ \min(Var(g), Var(f)) \ , \tag{67}$$

which achieves the proof of the lemma. □

This regularity allows the use of a global minimization algorithm.

An algorithm to find the optimal translation.

In order to find the optimal translation between two L^1 functions, we give a global optimization procedure, relying only upon the Lipschitz regularity (stated in the previous section) of the translation operator with respect to the K-norms. Assuming that the optimal translation τ_0 is searched in the set $[0;1]$, the algorithm writes as follows :

- Initialization. Let $n_0 \in \mathbb{N}$. Compute $C(\frac{k}{2^{n_0}})$ for $k \in [0; 2^{n_0}]$. Denote by $(I_0^0, I_1^0, ..., I_{2^{n_0}-1}^0)$ the sequence of intervals.
- Step 1 : reduction of the search area. Let $(I_0^k, I_1^k, ..., I_{n_k}^k)$ be the current sequence of intervals. Using the Lipschitz constant, it is possible to estimate the value of the cost on the whole intervals. Accordingly, we have an estimation of the minimum value of C. The Lipschitz regularity allow to check if a global minimum can be located in any of the intervals. Hence we can reduce the search area.
- Step 2 : refining of the "step-length". Compute the value of the cost at the midpoint of each of the remaining intervals, defining two subintervals. Then go back to step 1 with the new sequence of intervals.

As a conclusion, we can observe that K-norms lead to a smoother optimization problem than the total variation. Moreover, it seems that they reduce oscillations, in the sense that there are fewer local extremas.

6 Conclusion

In this paper, we pointed out the potential interest of Extended Kantorovich norms in optimization. Indeed, those norms allow to consider a wider class of functions than usual metrics deriving from optimal transport problems, while keeping a valuable property : K-norms metrize the weak* convergence for measures (see theorem 2.4). In the mono-dimensional case (or for Monge cost matrices), we stated a greedy property, which is proved to be useful in the computation of K-norms.

At this point, we have to mention that K-norms are not differentiable. However, this drawback can be balanced by the fact that K-norms may lead to smoother problems, as in the case of the translation problem (see section 5). Finally, it remains to find real applications where K-norms would prove their efficiency and overcome the non-differentiability drawback. This should be the purpose of future works.

Appendix A : detailed proof of theorem 4.7

Our aim is to prove that $I = S_0 + S_2 - 2S_1$ is non-negative. Up to a transposition, we can assume that $i_1 \leq j_1$. We consider several cases.

- First case : $i_1 < i_2 \leq j_2 < j_1$.
A careful arrangement of the terms in I leads to the expression $I = I_1 + I_2 + I_3$, where

$$I_1 = \sum_{k=i_2}^{j_2} (c[k+1,k-1] + c[k,k]) - (c[k,k-1] + c[k+1,k]) \ ,$$
$$I_2 = \sum_{k=i_1}^{i_2-1} (c[k,k] - c[k+1,k]) \ ,$$
$$I_3 = \sum_{k=j_2+1}^{j_1} (c[k,k] - c[k,k-1]) \ .$$

Using the quadrangle inequality, we get $I_1 \geq 0$. The last two inequalities follow from the "optimality" of C_1. Indeed, i_1 (resp. j_1) is a better single-person than i_2 (resp. j_2). This implies that $I_1 \geq 0$ (resp. $I_3 \geq 0$). Hence we have $I \geq 0$.

- Second case : $i_1 < i_2 \leq j_1 < j_2$.
A careful arrangement of the terms in I leads to the expression $I = I_1 + I_2 + I_3$, where

$$I_1 = \sum_{k=i_2}^{j_1} (c[k,k] + c[k+1,k-1]) - (c[k+1,k] + c[k,k-1]) \ ,$$

$$I_2 = \sum_{k=i_1}^{i_2-1} c[k,k] - c[k+1,k] \ ,$$

$$I_3 = \sum_{k=j_1+1}^{j_2} c[k,k-1] - c[k,k] \ .$$

Using the quadrangle inequality, we deduce that $I_1 \geq 0$. As in the first case, the last two inequalities follow from the "optimality" of C_1. Since i_1 (resp. j_1) is a better single-person than i_2 (resp. j_2), we get $I_2 \geq 0$ (resp. $I_3 \geq 0$). We deduce that $I \geq 0$.

- Third case : $i_2 < i_1 \leq j_1 < j_2$.

A careful arrangement of the terms in I leads to the expression $I = I_1 + I_2 + I_3$, where

$$I_1 = \sum_{k=i_1}^{j_1} (c[k,k] + c[k+1,k-1]) - (c[k+1,k] + c[k,k-1]) \; ,$$

$$I_2 = \sum_{k=i_2}^{i_1-1} c[k+1,k] - c[k,k] \; ,$$

$$I_3 = \sum_{k=j_1+1}^{j_2} c[k,k-1] - c[k,k] \; .$$

Using the quadrangle inequality, we deduce that $I_1 \geq 0$. As in the first case, the last two inequalities follow from the "optimality" of C_1. Since i_1 (resp. j_1) is a better single-person than i_2 (resp. j_2), we get $I_2 \geq 0$ (resp. $I_3 \geq 0$). We deduce that $I \geq 0$.

- Fourth case : $i_1 < j_2 < i_2 < j_1$.
  Since $C_1 = (i_1, j_1)$ is better than (i_1, j_2) and (i_2, j_1), we have the following inequalities

$$S_1 \leq \sum_{k=i_1}^{j_2-1} c[k+1,k] + \sum_{k=j_2}^{j_1-1} c[k+1,k+1] \; ,$$

$$S_1 \leq \sum_{k=i_1}^{i_2-1} c[k,k] + \sum_{k=i_2}^{j_1-1} c[k+1,k] \; .$$

  We deduce that

$$\begin{aligned} 2S_1 &\leq \sum_{k=i_1}^{j_2-1} c[k+1,k] + \sum_{k=j_2}^{j_1-1} c[k+1,k+1] + \sum_{k=i_1}^{i_2-1} c[k,k] + \sum_{k=i_2}^{j_1-1} c[k+1,k] \; , \\ &\leq \left(\sum_{k=i_1}^{j_1} c[k,k] \right) + \left(\sum_{k=i_1}^{j_2-1} c[k+1,k] + \sum_{k=j_2+1}^{i_2-1} c[k,k] + \sum_{k=i_2}^{j_1-1} c[k+1,k] \right) . \end{aligned}$$

  On the right side, we recognize S_0 and S_2. Then we get $2S_1 \leq S_0 + S_2$, which leads to $I \geq 0$.

- Fifth case : $i_1 \leq j_2 < j_1 \leq i_2$.
  Since $C_1 = (i_1, j_1)$ is better than (i_1, j_2) and (i_2, j_1), we have the following inequalities

$$S_1 \leq \sum_{k=i_1}^{j_2-1} c[k+1,k] + \sum_{k=j_2+1}^{i_2} c[k,k] \; ,$$

$$S_1 \leq \sum_{k=i_1}^{j_1-1} c[k,k] + \sum_{k=j_1}^{i_2-1} c[k,k+1] \; .$$

  We deduce that

$$\begin{aligned} 2S_1 &\leq \sum_{k=i_1}^{j_1-1} c[k,k] + \sum_{k=j_2+1}^{i_2} c[k,k] + \sum_{k=i_1}^{j_2-1} c[k+1,k] + \sum_{k=j_1}^{i_2-1} c[k,k+1] \; , \\ &\leq \left(\sum_{k=i_1}^{j_1} c[k,k] \right) + \left(\sum_{k=i_1}^{j_2-1} c[k+1,k] + \sum_{k=j_2+1}^{j_1-1} c[k,k] + \sum_{k=j_1}^{i_2-1} c[k,k+1] \right) . \end{aligned}$$

  On the right side, we recognize S_0 and S_2. Then we get $2S_1 \leq S_0 + S_2$, which leads to $I \geq 0$.

- Sixth case : $j_2 < i_1 \leq j_1 < i_2$.
Since $C_1 = (i_1, j_1)$ is better than (i_1, j_2) and (i_2, j_1), we get the following inequalities

$$S_1 \leq \sum_{k=j_2}^{i_1-1} c[k,k+1] + \sum_{k=i_1+1}^{i_2} c[k,k] \; ,$$

$$S_1 \leq \sum_{k=j_2}^{j_1-1} c[k,k] + \sum_{k=j_1}^{i_2-1} c[k,k+1] \; .$$

We deduce that

$$\begin{aligned} 2S_1 &\leq \sum_{k=j_2}^{i_1-1} c[k,k+1] + \sum_{k=j_1}^{i_2-1} c[k,k+1] + \sum_{k=j_2}^{j_1-1} c[k,k] + \sum_{k=i_1+1}^{i_2} c[k,k] \; , \\ &\leq \left(\sum_{k=j_2}^{i_2} c[k,k] \right) + \left(\sum_{k=j_2}^{i_1-1} c[k,k+1] + \sum_{k=i_1+1}^{j_1-1} c[k,k] + \sum_{k=j_1}^{i_2-1} c[k,k+1] \right) . \end{aligned}$$

On the right side, we recognize S_0 and S_2. Then we get $2S_1 \leq S_0 + S_2$, which leads to $I \geq 0$.

This achieves the detailed proof of theorem 4.7.

Appendix B : total variation is a limit of K-norms

In this appendix, for any $p \in \mathbb{R}_+^*$, we denote by N_p the application

$$N_p : \mu \mapsto \inf_{\mu_0 \in \mathcal{M}'(\mathbb{R}^d)} p\|\mu_0\|_d^0 + Var(\mu - \mu_0) \; . \tag{68}$$

Then the following lemma holds.

Lemma 6.1. *Let $f \in \mathcal{M}(\mathbb{R}^d)$. Then $\lim_{p=\infty} N_p(f) = Var(f)$.*

Proof. It is easy to show that $p \mapsto N_p(f)$ is non-decreasing, and that

$$\forall p \in \mathbb{R}_+^*, \; N_p(f) \leq Var(f) \; . \tag{69}$$

Hence $N_p(f)$ converges as p goes to infinity. Now we prove that the limit is $Var(f)$. Let $\epsilon \in \mathbb{R}_+^*$. For any p, there exists some μ_p such that

$$\begin{aligned} p\|f - \mu_p\|_d^0 + Var(\mu_p) &\leq N_p(f) + \epsilon \; , & (70) \\ &\leq Var(f) + \epsilon \; . & (71) \end{aligned}$$

We deduce from this inequality that $\|f - \mu_p\|_d^0$ goes to 0 as p goes to infinity. Since the Kantorovich norm metrizes the weak-$*$ convergence of measures, we deduce that there exists some p_0 such that

$$Var(\mu_{p_0}) \geq Var(f) - \epsilon \; . \tag{72}$$

We deduce that $N_{p_0}(f) \geq Var(f) - 2\epsilon$. Then for any $p \geq p_0$, we get

$$Var(f) - 2\epsilon \leq N_p(f) \leq Var(f) \; . \tag{73}$$

This achieves the proof of the lemma. $\square$

Appendix C : a concavity property of K-norms

Lemma 6.2. *Let $f \in \mathcal{M}(\mathbb{R})$. Then the application*

$$\psi : p \mapsto \|f\|_p \ , \tag{74}$$

is continuous, non-decreasing and concave.

Proof. It is easy to prove that ψ is non-decreasing. Up to a density argument, it is possible to assume that f is a finite weighted sum of Dirac functions. Now let p_0 and p_1 be two positive numbers such that $p_0 \leq p_1$. Using theorem 4.6, we know that there exists a real constant K (corresponding to the cost of an allocation minimizing the waste of goods) and three integer $n_0 \leq n_{1/2} \leq n_1$ such that

$$\begin{aligned}
\psi(p_0) &= K + \sum_{i=0}^{n_0} (a_i + 2p_0)\ \delta_i \ , && (75)\\
\psi(p_1) &= K + \sum_{i=0}^{n_1} (a_i + 2p_1)\ \delta_i \ , && (76)\\
\psi\left(\frac{p_0+p_1}{2}\right) &= K + \sum_{i=0}^{n_{1/2}} (a_i + p_0 + p_1)\ \delta_i \ , && (77)
\end{aligned}$$

where (a_i) stands for the sequence of shift costs and (δ_i) stands for the sequence of shift amplitudes. Moreover, theorem 4.7 implies that the sequence (a_i) is non-decreasing, and satisfies the following property :

$$i \leq n_p \implies a_i \leq 2p \ . \tag{78}$$

Expanding (77), we get

$$\begin{aligned}
\psi\left(\frac{p_0+p_1}{2}\right) &= K + \sum_{i=0}^{n_{1/2}} (a_i + p_0 + p_1)\ \delta_i \ , && (79)\\
&= \frac{1}{2}\psi(p_1) + \frac{1}{2}\left(K + \sum_{i=0}^{n_1} (a_i + 2p_0)\ \delta_i\right) + \sum_{i=n_1+1}^{n_{1/2}} (a_i + p_0 + p_1)\ \delta_i \ , && (80)\\
&= \frac{\psi(p_0) + \psi(p_1)}{2} + S \ , && (81)
\end{aligned}$$

with $S = 2 \sum_{i=n_1+1}^{n_{1/2}} (a_i + p_0 + p_1)\ \delta_i - \sum_{i=n_1+1}^{n_0} (a_i + 2p_0)\ \delta_i$.
It remains to prove that $S \geq 0$. We have

$$\begin{aligned}
S &= \sum_{i=n_1+1}^{n_{1/2}} (p_1 - p_0)\ \delta_i + \sum_{i=n_1+1}^{n_{1/2}} (a_i + p_0 + p_1)\ \delta_i - \sum_{i=n_{1/2}+1}^{n_0} (a_i + 2p_0)\ \delta_i \ , && (82)\\
&= \sum_{i=n_1+1}^{n_{1/2}} (a_i + 2p_1)\ \delta_i - \sum_{i=n_{1/2}+1}^{n_0} (a_i + 2p_0)\ \delta_i \ . && (83)
\end{aligned}$$

Then (78) implies $S \geq 0$, and we get

$$\psi\left(\frac{p_0+p_1}{2}\right) \geq \frac{\psi(p_0) + \psi(p_1)}{2} \ . \tag{84}$$

Using the monotonicity of ψ and inequality (84), we deduce that ψ is continuous and concave. This achieves the proof of lemma 6.2. □

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